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INVERSE PROBLEM OF DEFORMATION OF A PHYSICALLY NONLINEAR INHOMOGENEOUS MEDIUM

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An isotropic linear-elastic (viscoelastic) plane containing various physically nonlinear elliptic inclusions is considered. It is assumed that the distances between the centers of the inclusions are much greater than their dimensions. The problem is to determine the orientation of the inclusions and the loads applied at infinity which ensure a specified value of the principal shear stress in each inclusion. Necessary and sufficient conditions of existence of the solution of the problem are formulated for a plane strain of an incompressible inhomogeneous medium.

We consider an isotropic linear-elastic plane with physically nonlinear elliptic inclusions (PNEI) with different mechanical properties, dimensions, and orientations of symmetry axes. In the kth PNEI denoted by S_k^*, we choose the coordinate system $O_k x_{1k} x_{2k}$ in such a manner that the equation of the boundary L_k separating S_k^* from the elastic medium S has the form $x_{1k}^2 a_k^{-2} + x_{2k}^2 b_k^{-2} = 1$, where $a_k \geqslant b_k$ (hereafter, summation over k is not performed).

Let the distance between the centers of two arbitrary PNEI be much greater than their dimensions: $|\overline{O_k O_l}| \gg \max\limits_i a_i \ \forall\ k, l$. In this case, the interaction between the stress-strain states of inclusions can be ignored.

Let uniformly distributed stresses act at infinity. We denote the principal values of the stresses by N_1 and N_2 and the angle between the first principal axis and the $O_k x_{1k}$-axis by α_k.

We assume that the entire region $S \cup S_k^*$ $(k = 1, 2, \ldots)$ undergoes plane strain and the elastic medium and all PNEI are incompressible. Hence, in any coordinate system Ox_1x_2 in S, the strains ε_{ij} are related to the stresses σ_{ij} $(i, j = 1, 2)$ by the formulas [1]

$$4\mu\varepsilon_{22} = -4\mu\varepsilon_{11} = \sigma_{22} - \sigma_{11}, \qquad 2\mu\varepsilon_{12} = \sigma_{12}, \tag{1}$$

where μ is the shear modulus. [We note that, if μ is replaced by $\mu(1 + K)$ (K is the Volterra operator [2]), relations (1) correspond to plane strain of a linear viscoelastic incompressible medium.]

We assume that the kth inclusion is isotropic and nonlinear elastic (or obeys the deformation theory of plasticity). In this case, the constitutive equations in the coordinate system $O_k x_{1k} x_{2k}$ have the form

$$\begin{gathered} \varepsilon_{22k}^* = -\varepsilon_{11k}^* = F_k(\tau_k^*)(\sigma_{22k}^* - \sigma_{11k}^*)/2, \\ \varepsilon_{12k}^* = F_k(\tau_k^*)\sigma_{12k}^*, \quad 2\tau_k^* = [(\sigma_{22k}^* - \sigma_{11k}^*)^2 + 4\sigma_{12k}^{*2}]^{1/2} \quad (k = 1, 2, \ldots). \end{gathered} \tag{2}$$

Here $F_k(\tau_k^*) > 0$ is a specified function and τ_k^* is the principal shear stress. (Relations (2) can be complicated by replacing their right sides by nonlinear operators [1, 3].) As in [1, 3], we assume that the strains of the medium and PNEI are small and the load and displacement fields are continuous on the boundaries L_k $(k = 1, 2, \ldots)$.

Since, by assumption, inclusions do not interact with one another, we use the following relations between the stress-strain state (uniform in this case) of the kth PNEI (in the coordinate system $O_k x_{1k} x_{2k}$) and the loads at infinity [1, 3]:

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$$\mu(m_k\bar{C}_k + \bar{D}_k) = m_kA_k + B_k - 2(m_k\Gamma + \Gamma'_k), \quad \mu(\bar{C}_k + m_k\bar{D}_k) = -(A_k + m_kB_k) + 2\Gamma,$$

$$2A_k = \sigma^*_{11k} + \sigma^*_{22k}, \quad 2B_k = \sigma^*_{22k} - \sigma^*_{11k} + 2i\sigma^*_{12k}, \quad C_k = \varepsilon^*_{11k} + \varepsilon^*_{22k} + 2i\varepsilon^*_k,$$

$$D_k = \varepsilon^*_{11k} - \varepsilon^*_{22k} + 2i\varepsilon^*_{12k}, \quad m_k = (a_k - b_k)/(a_k + b_k), \tag{3}$$

$$4\Gamma = N_1 + N_2, \quad \Gamma'_k = \Gamma'_0 \,\mathrm{e}^{-2i\alpha_k}, \quad 2\Gamma'_0 = N_2 - N_1 \quad (k = 1, 2, \ldots).$$

Here ε^*_k is the rotation in S^*_k; the rotation at infinity is $\varepsilon^\infty = 0$.

The inverse problem is formulated as follows. Is it possible (and under which conditions) to choose the loads N_1 and N_2 (the principal directions are assumed to be known) and the angles α_k so that the principal shear stress in each inclusion takes a specified value, i.e., the equalities $\tau^*_k = \tau_{0k}$ hold (τ_{0k} are the specified values and $k = 1, 2, \ldots$)?

We show that, under certain restrictions, the solution of the above problem exists. Taking into account that the equalities $|B_k| = \tau^*_k$, $C_k = 2i\varepsilon^*_k$, and $\bar{D}_k = -2F_k(\tau^*_k)B_k$ hold by virtue of (2) and (3) and setting $B_k = \tau_{0k}\,\mathrm{e}^{i\varphi_k}$, from (3) we obtain

$$2\Gamma'_0\,\mathrm{e}^{-2i\alpha_k} = [(1 - m_k^2) + \beta_k(1 + m_k^2)]\tau_{0k}\,\mathrm{e}^{i\varphi_k} + 4i\mu m_k\varepsilon^*_k,$$

$$2\Gamma = A_k + m_k(1 - \beta_k)\tau_{0k}\,\mathrm{e}^{i\varphi_k} - 2i\mu\varepsilon^*_k, \qquad \beta_k = 2\mu F_k(\tau_{0k}). \tag{4}$$

Since Γ and A_k are real quantities, the second relation of (4) implies

$$2\mu\varepsilon^*_k = m_k(1 - \beta_k)\tau_{0k}\sin\varphi_k. \tag{5}$$

Substitution of (5) into (4) yields

$$2\Gamma'_0\cos 2\alpha_k = [(1 - m_k^2) + \beta_k(1 + m_k^2)]\tau_{0k}\cos\varphi_k,$$

$$-2\Gamma'_0\sin 2\alpha_k = [(1 + m_k^2) + \beta_k(1 - m_k^2)]\tau_{0k}\sin\varphi_k, \tag{6}$$

which can be written in a more convenient form

$$2\Gamma'_0\tau_{0k}^{-1}\,\mathrm{e}^{-2i\alpha_k} = (1 + \beta_k)\,\mathrm{e}^{i\varphi_k} - m_k^2(1 - \beta_k)\,\mathrm{e}^{-i\varphi_k}.$$

Multiplying this equality by the conjugate equality, i.e., eliminating α_k from (6), we obtain

$$(2\Gamma'_0\tau_{0k}^{-1})^2 = (1 + \beta_k)^2 - 2m_k^2(1 + \beta_k)(1 - \beta_k)\cos 2\varphi_k + m_k^4(1 - \beta_k)^2. \tag{7}$$

From (7) follows

$$\cos 2\varphi_k = [(1 + \beta_k)^2 + m_k^4(1 - \beta_k)^2 - (2\Gamma'_0\tau_{0k}^{-1})^2]/[2m_k^2(1 + \beta_k)(1 - \beta_k)]. \tag{8}$$

Equality (8) is valid if the absolute value of its right side does not exceed unity. Solving the corresponding inequalities and taking into account that $1 + \beta_k > m_k^2|1 - \beta_k|$ [since (3) and (4) imply that $m_k^2 < 1$ and $\beta_k > 0$], we obtain

$$F_{1k}(\tau_{0k}) \leqslant 2|\Gamma'_0| \leqslant F_{2k}(\tau_{0k}),$$

$$F_{1k} \equiv (1 + \beta_k - m_k^2|1 - \beta_k|)\tau_{0k}, \quad F_{2k} \equiv (1 + \beta_k + m_k^2|1 - \beta_k|)\tau_{0k}. \tag{9}$$

Inequalities (9) are satisfied for any k under the necessary and sufficient conditions

$$\max_k F_{1k} \leqslant 2|\Gamma'_0| \leqslant \min_k F_{2k}. \tag{10}$$

It follows from (10) that the solution of the problem exists if

$$\max_k F_{1k}(\tau_{0k}) \leqslant \min_k F_{2k}(\tau_{0k}). \tag{11}$$

If (11) is satisfied, Γ'_0 can take any value within the interval determined by inequalities (10), and the angle φ_k is found from (8). In the interval $[-\pi, \pi]$, this angle can take four values with different signs which differ by the quantity $\pm\pi$. Given Γ'_0 and φ_k, the angle α_k is found from (6) (each value of φ_k in the same interval corresponds to two values of α_k which differ by π). The quantity Γ can be written as $\Gamma = \Gamma_0$ (Γ_0 is an arbitrary constant). In this case, A_k is determined from the second equality of (4):

$$A_k = 2\Gamma_0 - m_k(1 - \beta_k)\tau_{0k}\cos\varphi_k.$$

One can easily show that, for specified values of Γ_0' and α_k, the quantities τ_k and φ_k are determined uniquely, i.e., the values of $\tau_k^* = \tau_{0k}$ corresponds to the determined quantities Γ_0' and α_k. To this end, it is sufficient to establish that system (6) is uniquely solvable for τ_{0k} and φ_k ($-\pi \leqslant \varphi_k \leqslant \pi$). In this case, it is assumed that the constitutive equations (2) for PNEI satisfy the stability conditions [4]

$$\Delta\sigma_{ijk}^* \Delta\varepsilon_{ijk}^* \geqslant 0$$

(summation is performed from 1 to 2 over i and j, and summation over k is not performed), which are reduced here to the inequalities [4, p. 129] $[\tau F_k(\tau)]' \geqslant 0$, i.e.,

$$[\tau\beta_k(\tau)]' \geqslant 0 \qquad (12)$$

(the prime denotes differentiation with respect to τ).

From (6), we obtain

$$(2\Gamma_0')^{-2} = f(\tau_{0k}) \equiv \frac{\cos^2 2\alpha_k}{[(1-m_k^2)\tau_{0k} + (1+m_k^2)\tau_{0k}\beta_k]^2} + \frac{\sin^2 2\alpha_k}{[(1+m_k^2)\tau_{0k} + (1-m_k^2)\tau_{0k}\beta_k]^2}.$$

Hence, with allowance for (12) and inequalities $m_k^2 < 1$ and $\beta_k > 0$, we have $f'(\tau_{0k}) < 0$. It follows that an inverse single-valued function $\tau_{0k} = \tau_{0k}(\Gamma_0')$ exists. For known values of τ_{0k}, the quantities $\cos\varphi_k$ and $\sin\varphi_k$ are uniquely determined from (6). The statement is proved.

Condition (11) imposes stringent restrictions on the quantities τ_{0k}. We consider a particular case where inequality (11) is satisfied. Let all the PNEI have identical mechanical properties, i.e., all $F_k = F$ in (2), and it is required to choose the stresses N_1 and N_2 at infinity so that the quantity τ_k^* is the same in all the inclusions: $\tau_k^* = \tau_0$. In this case, $\beta_k = \beta_0 \equiv 2\mu F(\tau_0)$, condition (11) holds, and Γ_0' can be written, for example, in the form $\Gamma_0' = (1+\beta_0)\tau_0/2$. Hence, by virtue of (8), we obtain

$$\cos 2\varphi_k = m_k^2(1-\beta_0)/(2(1+\beta_0)),$$

which is valid for any τ since

$$|m_k^2(1-\beta_0)/(2(1+\beta_0))| < m_k^2/2 < 1/2,$$

inasmuch as $\beta_0 > 0$.

As was pointed out above, one can use more complex relations instead of (1) and (2): replace (1) by equations of a linear viscoelastic medium and replace (2) by equations of a nonlinear viscoelastoplastic inclusion (or an inclusion that exhibits creep properties or one that accumulates damages and fails because of creep). In this case, the quantities β_k in (4) and all the subsequent formulas are replaced by the Volterra operators. For these media, one can formulate a problem similar to that considered above, i.e., the problem of optimal deformation with time and fracture of a PNEI.

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REFERENCES

1. I. Yu. Tsvelodub, "Determination of the strength characteristics of a physically nonlinear inclusion in a linearly elastic medium," *J. Appl. Mech. Tech. Phys.*, **41**, No. 4, 734–739 (2000).
2. Yu. N. Rabotnov, *Elements of Heritable Mechanics of Solids* [in Russian], Nauka, Moscow (1977).
3. I. Yu. Tsvelodub, "Physically nonlinear inclusion in a linear-elastic medium (plane problem)," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 5, 72–84 (2000).
4. I. Yu. Tsvelodub, *Stability Postulate and Its Application to the Theory of Creep of Metals* [in Russian], Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1991).